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Exam one: MTH 221, Spring 2017

Ayman Badawi

SCORE = $\frac{\quad}{69}$

QUESTION 1. (3 points) Let A be a 3×3 matrix such that $A \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix} = 5 \begin{bmatrix} 16 \\ 12 \\ -8 \end{bmatrix}$. Now stare. I claim you should see one eigenvalue of A, say α. What is α? explain

Solution. By staring, we observe that $A \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix} = 5 \times 4 \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix}$**. Hence 20 is an eigenvalue of A**

QUESTION 2. (10 points) Let A be a 3×3 matrix such that 3, 2 are the eiginvalues of A, where $Trace(A) = 8$, $E_3 = span\{(1,1,1)\}$ and $E_2 = span\{(0,1,4)\}$.

(i) Is A diagnolizable? if no , then explain. if yes, then find an invertible matrix Q and a diagonal matrix D such that $QDQ^{-1} = A$.

Solution. Note $C_A(\alpha)$ **is of degree 3. So it cannot have more than 3 roots.** $8 = Trace(A) = 3 + 2 + a$**. Hence** $a = 3$**. Thus** 3 **is repeated twice but** E_3 **is a span of one point. Thus** A **is not diagnolizable,**

(ii) Let $B = 3I_3 - A$. Find the solution set to the system of linear equations $B \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Solution. Note that the solution set to the (homogenous) system of the linear equations $B \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ **is** $Z(B) = N(B) = E_3 = Span\{(1,1,1)\}$**.**

(iii) Let $B = 5I_3 - A$. Find the solution set to the system of linear equations $B \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Solution. Since 5 is not an eigenvalue of A**, we conclude that the solution set to** $Z(B) = Z(5I_3 - A) = \{(0,0,0)\}$.

(iv) Find $|A|$.

Solution. $|A| = 3 \times 3 \times 2 = 18$.

(v) Find $|A + 2I_3|$

Solution. Eigenvalues of $A + 2I_3$ **are (3 + 2)= 5, (3 + 2)= 5, (2 + 2) = 4. Hence** $|A + 2I_3| = 5 \times 5 \times 4 = 100$

QUESTION 3. (**4 points**) For what values of a, b, c that make the matrix $A = \begin{bmatrix} 3 & a & b \\ 0 & 3 & c \\ 0 & 0 & 3 \end{bmatrix}$ diagnolizable? EXPLAIN.

Solution. $C_A(\alpha) = det(\begin{bmatrix} \alpha - 3 & -a & -b \\ 0 & \alpha - 3 & -c \\ 0 & 0 & \alpha - 3 \end{bmatrix} = (\alpha - 3)^3$. **Hence** 3 **is an eigenvalue of** A **repeated 3 times. So** $E_3 = Z\left(\begin{bmatrix} 0 & -a & -b \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{bmatrix}\right)$ **must be a span of 3 independent points = number of free variables. Thus the matrix** $\begin{bmatrix} 0 & -a & -b \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{bmatrix}$ **cannot have a nonzero-row (if it has a nonzero row then we can have a leading variable, and hence number of free variables will be strictly less than 3). Thus** $a = b = c = 0$.

QUESTION 4. (**4 points**) Let A be a matrix and $(A^T)^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 4 \\ -1 & -2 & 10 \end{bmatrix}$. Find the solution set to $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Solution: Note that $(A^T)^{-1} = (A^{-1})^T$. **Thus** $A^{-1} = ((A^{-1})^T)^T = ((A^T)^{-1})^T = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 2 & -2 \\ 3 & 4 & 10 \end{bmatrix}$. **Hence** $A^{-1}A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 2 & -2 \\ 3 & 4 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. **Thus** $I_3 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ -2 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 13 \end{bmatrix}$. **Thus the solution set =** $\{(0, 0, 13)\}$.

QUESTION 5. (**5 points**) Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & -2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$. Find A^{-1} if possible

Solution: $\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 + R_2 \to R_2}, \xrightarrow{-2R_1 + R_3 \to R_3} \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{bmatrix}$ $\xrightarrow{0.5R_2} \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0.5 & 0.5 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 + R_2 \to R_2} \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1.5 & 0.5 & 1 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{bmatrix}$ **Now stare and make the left hand matrix equals** I_3 **by interchanging rows only. Hence** $A^{-1} = \begin{bmatrix} -1.5 & 0.5 & 1 \\ -2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

QUESTION 6. Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$ and $B = \begin{bmatrix} -2a_7 & -2a_8 & -2a_9 \\ 7a_4 & 7a_5 & 7a_6 \\ 5a_4 + a_1 & 5a_5 + a_2 & 5a_6 + a_3 \end{bmatrix}$. If $|A| = -4$, then

a) Find $|B|$ (**4 points**)

Solution: We start with the given A : $A \ \overrightarrow{5R_2 + R_1 \to R_1} \ C \ \overrightarrow{R_1 \leftrightarrow R_3} \ D \ \overrightarrow{-2R_1, 7R_2} \ B$ **is the given B. Hence** $|B| = -1 \times -2 \times 7 \times |A| = -1 \times -2 \times 7 \times -4 = -56$

b) Find $|-2A^{-1}|$(**3 points**)

Solution: $|-2A^{-1}| = \frac{(-2)^3}{-4} = 2$

QUESTION 7. Given $A = \begin{bmatrix} d & a \\ b & c \end{bmatrix}$ such that a, b are NONZERO INTEGERS and $A^{-1} = A$.

a) Find $|A|$. (**3 points**)

Solution: Since A^{-1} **exists,** $|A| \neq 0$**. Hence** $\frac{1}{|A|} = |A|$**. Thus** $(|A|)^2 = 1$**. Thus** $|A| = 1$ **or** $|A| = -1$**. Assume** $|A| = 1$**. Hence** $A^{-1} = \begin{bmatrix} c & -a \\ -b & d \end{bmatrix} = A = \begin{bmatrix} d & a \\ b & c \end{bmatrix}$**. Thus** $-a = a$ **and hence** $2a = 0$ **which implies that** $a = 0$**, impossible since** a **is a nonzero integer (given in the question). Thus** $|A| \neq 1$ **and hence** $|A| = -1$

b) Give me one possibility of A .(**2 points**)

Solution: by (a) $|A| = -1$**. Hence** $A^{-1} = \begin{bmatrix} -c & a \\ b & -d \end{bmatrix} = A = \begin{bmatrix} d & a \\ b & c \end{bmatrix}$**. Hence** $c = -d$**,** a, b **any nonzero integers such that** $|A| = cd - ab = -c^2 - ab = -1$**. Choose** $c = 0$**, then** $d = 0$**. Hence** $|A| = -ab = -1$**. So choose** $a = b = 1$ **or** $a = b = -1$**. So we get** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **or** $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ **other examples** $\begin{bmatrix} -6 & 7 \\ -5 & 6 \end{bmatrix}, \begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix}$

QUESTION 8. (**6 points**) Find a matrix A, 2×3, such that $\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} A + A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

Solution: $\left(\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} + I_2 \right) A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$**. Hence** $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$**. Thus** $\left(\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} A = \left(\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$**. Thus** $I_2 A = A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

QUESTION 9. (**6 points**) Find the solution set to the system

$$x_1 + 2x_4 + x_5 = 4$$
$$x_2 + x_3 - x_5 = 8$$
$$-x_1 - x_4 - x_5 = 2$$

Solution: we start with the augmented matrix

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 1 & 4 \\ 0 & 1 & 1 & 0 & -1 & 8 \\ -1 & 0 & 0 & -1 & -1 & 2 \end{bmatrix} \xrightarrow[R_1 + R_3 \to R_3]{} \begin{bmatrix} 1 & 0 & 0 & 2 & 1 & 4 \\ 0 & 1 & 1 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 & 0 & 6 \end{bmatrix} \xrightarrow[-2R_3 + R_1 \to R_1]{}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -8 \\ 0 & 1 & 1 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 & 0 & 6 \end{bmatrix}$$ **Now read:** $x_1 = -8 - x_5, x_2 = 8 - x_3 + x_5, x_4 = 6, x_3$ **and** $x_5 \in R$. **Hence the solution set is** $\{(-8 - x_5, 8 - x_3 + x_5, x_3, 6, x_5) \mid x_3, x_5 \in R\}$

QUESTION 10. Let $A = \begin{bmatrix} a & 0 & 0 \\ b & a & -4 \\ c & 1 & a-4 \end{bmatrix}$. Consider the system of linear equations $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2a-4 \\ a-2 \end{bmatrix}$ [Note $a - 2 = 2 + (a - 4)$]

a) For what values of a, b, c will the system have infinitely many solutions?(**4 points**)

Solution: First we show that the system is consistent: (i.e., find a solution). By staring and make use of the hint we observe

$$0 \begin{bmatrix} a \\ b \\ c \end{bmatrix} + 2 \begin{bmatrix} 0 \\ a \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ -4 \\ a-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2a-4 \\ a-2 \end{bmatrix}$$ **. Hence** $(0, 2, 1)$ **is a solution. Thus the system is consistent. Now in order to have infinitely many solution we must have** $|A| = 0$**. So** $|A|$ **(using the first row of** A**) =** $a(a^2 - 4a + 4)$**. Hence set** $a(a^2 - 4a + 4) = 0$**, we get** $a = 0$ **or** $a = 2$ **and** $b, c \in R$.

b) Give me one particular solution to the system. (**3 points**)

Solution: by (a) we have $(0, 2, 1)$ **is a solution to the system**.

QUESTION 11. Let $A = \begin{bmatrix} 1 & -4 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

a) Is A diagnolizable? If yes, then find an invertible matrix Q and a diagonal matrix D such that $QDQ^{-1} = A$.**(9 points)**

Solution: $C_A(\alpha) = |\alpha I_3 - A| = det\left(\begin{bmatrix} \alpha-1 & 4 & -8 \\ 0 & \alpha+1 & 0 \\ 0 & 0 & \alpha+1 \end{bmatrix}\right) = (\alpha-1)(\alpha+1)^2$. **We set** $C_A(\alpha) = (\alpha-1)(\alpha+1)^2 = 0$. **Hence** $\alpha = 1$, $alpha = -1$ **(repeated twice).**

Now $E_1 = Z(I_3 - A) = Z\left(\begin{bmatrix} 0 & 4 & -8 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right)$ **(i.e., the solution set to the homogenous system** $\begin{bmatrix} 0 & 4 & -8 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$**). We get** $x_2 = 0$**,** $x_3 = 0$**, and** $x_1 \in R$**. Hence** $E_1 = \{(x_1, 0, 0) \mid x_1 \in R\} = span\{(1,0,0)\}$**.**

$E_{-1} = Z(-I_3 - A) = \left(\begin{bmatrix} -2 & 4 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$**. We get** $x_1 = 2x_2 - 4x_3$**,** x_2 **and** $x_3 \in R$**. Thus** $E_{-1} = \{(2x_2 - 4x_3, x_2, x_3) \mid x_2, x_3 \in R\} = \{x_2(2,1,0) + x_3(-4,0,1) \mid x_2, x_3 \in R\} = span\{(2,1,0), (-4,0,1)\}$**.**

Now 1 is repeated once, E_1 **is span of one point, -1 is repeated twice, and** E_{-1} **is span of two independent points. Thus** A **is diagnolizable.**

Choose $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$**. Hence** $Q = \begin{bmatrix} 1 & 2 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$**. Thus** $Q\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}Q^{-1} = A$**.**

b) Find A^{2018} **(3 points)**

Solution: $A^{2018} = Q\begin{bmatrix} 1^{2018} & 0 & 0 \\ 0 & (-1)^{2018} & 0 \\ 0 & 0 & (-1)^{2018} \end{bmatrix}Q^{-1} = Q\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}Q^{-1} = QI_3Q^{-1} = QQ^{-1} = I_3$.

Faculty information

Ayman Badawi, Department of Mathematics & Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates.
E-mail: abadawi@aus.edu, www.ayman-badawi.com